Generalized Heun and Lamé's equations: factorization

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Abstract

This paper addresses new results on the factorization of the general Heun's operator, extending the investigations performed in previous works [*Applied Mathematics and Computation* **141** (2003), 177 - 184 and **189** (2007), 816 - 820]. Both generalized Heun and generalized Lamé equations are considered.

Key words: Factorization, Heun's differential equation, Lamé's differential equation, polynomials.

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1 Introduction

Fuchsian differential's equations of second order and their confluent forms, built for instance from the Ince techniques [1] play a major role in many partial differential equations of mathematical physics as Laplace, Helmohltz or Schrödinger's equations.

Without loss of generality the $k+3$ regular single singularities can be located at $x=0$, $x=1$, $x=a_i$, $i=\overline{1,\,k}$ and $x=\infty$ and after an appropriate change of function one of the indices at each finite singulary can be shifted to zero allowing to write the Fuchsian equation as [2,3]:

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$$y''(x) + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \sum_{i=1}^{k} \frac{\epsilon_i}{x-a_i}\right) y'(x) + \frac{\alpha\beta x^k + \sum_{i=1}^{k} \rho_i x^{k-i}}{x(x-1)\prod_{i=1}^{k}(x-a_i)} y(x) = 0. \tag{1}$$

The two indices at each singularity a_i are $(0, 1-\epsilon_i)$ and $(0, 1-\gamma)$, $(0, 1-\delta)$, (α, β) at $x = 0$, $x = 1$, and $x = \infty$, respectively, taking into account the Fuchsian relation:

$$\alpha + \beta + 1 = \gamma + \delta + \sum_{i=1}^{k} \epsilon_i. \tag{2}$$

$\epsilon_i \equiv 0$, $(i = \overline{1, k})$, gives the hypergeometric equation and $k = 1$ the Heun's equation [7]. The character *reducible* or *irreducible* of linear ordinary differential equation (O.D.E) is important in relation with the monodromy group [2]. We call reducible a second order O.D.E which admits a non trivial solution satisfying a first order linear equation, and otherwise irreducible. This definition is used by many authors [4,5,6] with or without supplementary conditions. It allows to give an easy criterion of reducibility of an equation with polynomial coefficients, from a factorized approach in the following way.

Rewriting (1) in a polynomial form with $D = \frac{\mathrm{d}}{\mathrm{d}x}$,

$$\begin{aligned}\mathcal{H}_k[y(x)] &\equiv \left[Q_{k+2}(x)D^2 + Q_{k+1}D + Q_k\right] y(x)\\ &= [L(x)D + M(x)]\left[\bar{L}(x)D + \bar{M}(x)\right] y(x),\end{aligned} \tag{3}$$

where Q_j are polynomials of degree j, possible (or not possible) identification of the polynomials L, $\bar{L}$, M, $\bar{M}$ with polynomials Q_{k+2}, Q_{k+1}, Q_k will give all possible cases of reducibility and, a contrario, criteria of irreducibility for the equation $\mathcal{H}_k[y(x)] = 0$.

This was recently done for the Heun's equation [8,9]

$$y''(x) + \left[\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right] y'(x) + \frac{\alpha\beta x - q}{x(x-1)(x-a)} y(x) = 0 \tag{4}$$

and also for the four confluent Heun's equations [10]. For the Heun's equation, this factorization is possible and generate 6 non trivial situations in accordance with the known $F-$homotopic transformations [11]. R. S Maier [12] noticed that for the Lamé equation ($k = 1$ with $\gamma = \delta = \epsilon = 1/2$)

$$y''(x) + \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-a}\right) y'(x) - \frac{l(l+1)x + 4q}{4x(x-1)(x-a)} y(x) = 0, \tag{5}$$

the factorization reduces only to the two cases $l = 1$, $l = \frac{1}{2}$. He also mentioned that for the general Lamé equation with $k + 3$ singularities, the factorization is not known.

Even if some algorithms exist in order to factorize any ODE with rational function coefficients [13,14,15,16], factorization of an arbitrary $k-$Heun equation is probably untractable, except for using numerical tools.

The aim of this work is to investigate the case $k = 2$ for both generalized Heun and generalized Lamé equations [17] which are also important in lattice statistics [18]. These equations have invested a large number of applications in physics. See [19] (and references therein) for a nice review on physical applications of these equations. To mention a few, retrieved from the indicated works, where for instance Heun's equations as well as their solutions impose their usefulness, it is worthy of attention to outline their importance in the description of quasi-modes of near extremal black branes, in the hyperspherical harmonics with applications in three-body systems, in the elaboration of a method of calculation of propagators for the case of a massive spin 3/2 field for arbitrary space-time dimensions and mass, in parametric resonance after inflation, as well as in the separation of variables for the Schrödinger equation in a large number of problems, typically for the radial coordinate, and in non-linear formulations involving Painleve type equations. A number of traditional equations of mathematical physics, as for instance the Lame, spheroidal wave, and Mathieu equations, are also particular cases of Heun equations.

Expansion of the equation (3) gives:

$$\begin{aligned}\mathcal{H}_k[y(x)] &= (LD + M)\left(\bar{L}D + \bar{M}\right) y(x) \\ &= \left\{L\bar{L}D^2 + \left[L\left(\bar{L}' + \bar{M}\right) + M\bar{L}\right] D + \left(L\bar{M}' + M\bar{M}\right)\right\} y(x)\end{aligned} \tag{6}$$

and generate the 3 basic relations:

$$\begin{cases} Q_{k+2} = L\bar{L}, \\ Q_{k+1} = L\left(\bar{L}' + \bar{M}\right) + M\bar{L}, \\ Q_k \quad = L\bar{M}' + M\bar{M}. \end{cases} \tag{7}$$

Computation and properties of the polynomials $M \equiv M(x)$, $\bar{M} \equiv \bar{M}(x)$, $L \equiv L(x)$, $\bar{L} \equiv \bar{L}(x)$ are explicitly given in several situations.

2 Preliminary remarks

1) $Q_{k+2}(x)$ contains $k+2$ linear and different factors which are all present in the product $L(x)\bar{L}(x)$ in several ways. For instance, if j factors appear in L and $k+2-j$ in $\bar{L}$, the number of decomposition is given by the classical binomial result:

$$\sum_{j=0}^{k+2} \binom{k+2}{j} = 2^{k+2}, \tag{8}$$

excluding the extremal cases

$$\begin{aligned} L(x) = Q_{k+2}(x),\ \bar{L} = 1;\ Q_k = 0, \\ \mathcal{H}_k = LD(D + \bar{M}) \end{aligned} \tag{9}$$

and

$$\begin{aligned} L(x) = 1,\ \bar{L} = Q_{k+2}(x),\ M = 0 \text{ (integrable case } \bar{M} = Q_{k+1} - Q'_{k+2}, \\ Q_k = \bar{M}'),\ \mathcal{H}_k = D(\bar{L}D + \bar{M}) \end{aligned} \tag{10}$$

which are irrelevant or trivial in this study.

Selecting a pair L, $\bar{L}$ inside the $2^{k+2} - 2$ possibilities, computation of the polynomials $M(x)$ and $\bar{M}(x)$ of degrees m and $\bar{m}$, respectively, is done in 2 steps.

i) The coefficients in $Q_{k+1}(x)$ and $Q_{k+2}(x)$ allow to linearly compute the coefficients of $M(x)$ and $\bar{M}(x)$, of number $m + \bar{m} \leq k$ from the second and third relations in (7).

ii) The third relation in (7) gives the value of the accessory parameters ρ_i, $(i = \overline{1,k})$ and the product $\alpha\beta$, using the polynomials M and $\bar{M}$ computed in the first step.

2) The general $k-$Lamé's equation is a $k-$ Heun's equation with $\gamma = \delta = \epsilon = 1/2$, $(i = \overline{1}, k)$. The indices at each finite singularity are equal to $(0, 1/2)$ and the Fuchsian relation (2) gives $\alpha + \beta = k/2$. This equation takes the form

$$\mathcal{L}_k[y] \equiv Q_{k+2}y'' + Q_{k+1}y' + \frac{1}{4}\left(\alpha\beta x^k + \sum_{i=1}^{k} \rho_i x^{k-i}\right) y = 0 \tag{11}$$

with now

$$Q_{k+1} = \frac{1}{2}(Q_{k+2})' = \frac{1}{2}\left[L\bar{L}\right]', \tag{12}$$

where

$$Q_{k+2} = x(x-1)\prod_{j=1}^{k}(x - a_j) = L\bar{L}. \tag{13}$$

The property (12) and second relation in (7) easily give:

$$\frac{1}{2}\left(\frac{L}{\bar{L}}\right)' = \frac{L\bar{M} + M\bar{L}}{\bar{L}^2}. \tag{14}$$

The number of situations to be investigated in the decomposition of $L\bar{L}$ can be simplified from some symmetry between polynomials M, $\bar{M}$ and $M^\star$, $\bar{M}^\star$ corresponding to the decomposition $L\bar{L}$, permuting L and $\bar{L}$ giving the factorization:

$$\mathcal{L}_k[y] = \left[\bar{L}D + M^\star\right]\left[LD + \bar{M}^\star\right] y(x). \tag{15}$$

A similar relation to (14) is

$$\frac{1}{2}\left(\frac{\bar{L}}{L}\right)' = \frac{\bar{L}\bar{M}^\star + M^\star L}{L^2}. \tag{16}$$

We deduce from (14) and (16) that

$$L\left(\bar{M} + M^\star\right) + \bar{L}\left(M + \bar{M}^\star\right) = 0. \tag{17}$$

Equality (17) is satisfied if, for example, we have $M = -\bar{M}^\star$, $\bar{M} = -M^\star$.

3 Lamé's equation, $(k = 1)$

Let us consider equation (4) with $\gamma = \delta = \epsilon = \frac{1}{2}$. Even if this case has been already solved in [7], as a subcase of the full Heun factorization, we easily recover the polynomials M and $\bar{M}$ from the procedure described in section 2 for the factorization $L = x(x-1)$, $\bar{L} = x-a$ as example. Unknown polynomials M and $\bar{M}$ must be as:

$$M = Ax + B,\ \bar{M} = \bar{A} \tag{18}$$

with

$$Q_3(x) = L\bar{L} = x^3 - x^2(a+1) + xa, \tag{19}$$

$$Q_2(x) = L(\bar{L}' + \bar{M}) + M\bar{L} = \frac{3}{2}x^2 - x(a+1) + \frac{a}{2} \tag{20}$$

$$Q_1(x) = M\bar{M}. \tag{21}$$

We easily conclude that

$$A = 1,\ B = -\frac{1}{2},\ \bar{A} = -\frac{1}{2}. \tag{22}$$

Next the third relation in (7) gives:

$$\alpha\beta = -\frac{1}{2},\ q = -\frac{1}{4} \tag{23}$$

and the Fuschian relation yields

$$\alpha + \beta = \frac{1}{2}. \tag{24}$$

From section 2, $M^\star = -\bar{M}$ and $\bar{M}^\star = -M$ already solve the factorization problem with $L = x - a$, $\bar{L} = x(x-1)$.

A surprising result, consequence of a remark of Maier [12], is that for $L = x$, or $(x-1)$, or $(x-a)$, $\big(\bar{L} = (x-1)(x-a),\ x(x-a),\ x(x-1)\big)$, the polynomials M are the same. From the symmetry property $M^\star = -\bar{M}$, $\bar{M}^\star = -M$, this result also works for the 3 other factorizations: $L = (x-1)(x-a)$, $x(x-a)$, $x(x-1)$; $\bar{L} = x$, $(x-1)$, $(x-a)$. All these results can be seen in table 1, presented in the same way as in Ref [9].

4 Factorization of Lame's equation $\mathcal{L}_2[y]$, $k = 2$

Equation (1) with $k = 2$, $a_1 = a$, $a_2 = b$ and $\gamma = \delta = \epsilon_1 = \epsilon_2 = \frac{1}{2}$ is the Lamé equation

$$\mathcal{L}_2[y] = Q_4(x)y'' + Q_3(x)y' + Q_2(x)y = 0, \tag{26}$$

where

$$\begin{cases} Q_4(x) = x(x-1)(x-a)(x-b) \\ \qquad\quad = x^4 - x^3(1+a+b) + x^2(a+b+ab) - abx, \\ Q_3(x) = \frac{1}{2}Q_4'(x) = \frac{1}{2}\left[4x^3 - 3x^2(1+a+b) + 2x(a+b+ab) - ab\right], \\ Q_2(x) = \alpha\beta x^2 + \rho_1 x + \rho_2, \\ \alpha + \beta = 1. \end{cases} \tag{27}$$

With a first choice $L = x(x-1)$, $\bar{L} = (x-a)(x-b)$, M and $\bar{M}$ are first degree polynomials

Table 1. Factorization of the simple Heun's operator $\mathcal{H}_1 = \mathcal{L}_1[y]$

$$
\begin{aligned}
\mathcal{H}_1 &= x(x-1)(x-a)D^2 + \left[\gamma(x-1)(x-a) + \delta x(x-a) + \epsilon x(x-1)\right] D + (\alpha\beta x - q)I_d \\
&= (LD+M)(\bar{L}D+\bar{M}),\ \gamma = \delta = \epsilon = \frac{1}{2}.
\end{aligned} \tag{25}
$$

L	$\bar{L}$	M	$\overline{M}$	α	β	q
x	$(x-1)(x-a)$	$\frac{1}{2}$	$-x+\frac{a+1}{2}$	$\frac{3}{2}$	-1	$-\frac{1}{2}\left(\frac{(a+1)}{2}\right)$
$x-1$	$x(x-a)$	$\frac{1}{2}$	$-x+\frac{a}{2}$	$\frac{3}{2}$	-1	$-\left(1+\frac{a}{4}\right)$
$x-a$	$x(x-1)$	$\frac{1}{2}$	$-x+\frac{1}{2}$	$\frac{3}{2}$	-1	$-\left(a+\frac{1}{4}\right)$
$(x-1)(x-a)$	x	$x-\frac{a+1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$-\frac{1}{4}(a+1)$
$x(x-a)$	$x-1$	$x-\frac{a}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$-\frac{1}{4}a$
$x(x-1)$	$x-a$	$x-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$-\frac{1}{4}$

$$M = Ax + B,\ \bar{M} = \bar{A}x + \bar{B}. \tag{28}$$

The four equations satisfied by $A,\ B,\ \bar{A},\ \bar{B}$ are given by identification of second relation in (7) with the second relation in (27). The solutions are

$$A = 1,\ B = -\frac{1}{2},\ \bar{A} = -1,\ \bar{B} = \frac{a+b}{2}. \tag{29}$$

Comparison of third relation in (7) with the third relation in (27) yields

$$\alpha\beta = -2 \tag{30}$$

and also ρ_1 and ρ_2:

$$\rho_1 = \frac{a+b+3}{2}, \tag{31}$$

$$\rho_2 = -\frac{a+b}{4}. \tag{32}$$

These results as well as those corresponding to the other choices of L and $\bar{L}$ are reproduced in tables 2 to 4 (14 situations) for the cases $\gamma = \delta = \epsilon_1 = \epsilon_2 = \frac{1}{2}$ as in the case $k = 1$, in which appear essentially 2 situations [12]; the case $k = 2$ generates only 3 situations with $\alpha\beta = -2,\ -3/4,\ -15/4$.

5 Factorization of the generalized Heun's equation $\mathcal{H}_2[y]$, $k = 2$

The first extension of Heun's equation with $k = 2$ can be written as $(y(x) \equiv y)$

$$y'' + \left[\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon_1}{x-a} + \frac{\epsilon_2}{x-b}\right] y' + \frac{\alpha\beta x^2 + \rho_1 x + \rho_2}{x(x-1)(x-a)(x-b)} y = 0 \tag{33}$$

with the parameters linked by the Fuchs's condition $\alpha + \beta + 1 = \gamma + \delta + \epsilon_1 + \epsilon_2$ or in polynomial way

$$\mathcal{H}_2[y] = Q_4(x)y'' + Q_3(x)y' + Q_2(x)y = 0, \tag{34}$$

where

$$\begin{aligned} Q_4(x) &= x(x-1)(x-a)(x-b) \\ &= x^4 - x^3(1+a+b) + x^2(a+b+ab) - xab; \end{aligned} \tag{35}$$

$$
\begin{aligned}
Q_3(x) &= \gamma(x-1)(x-a)(x-b) + \delta x(x-a)(x-b); \\
&\quad + \epsilon_1 x(x-1)(x-b) + \epsilon_2 x(x-1)(x-a) \qquad (36) \\
Q_2(x) &= \alpha\beta x^2 + \rho_1 x + \rho_2. \qquad (37)
\end{aligned}
$$

The factorized form is

$$
\begin{aligned}
\mathcal{H}_2[y] &= (LD+M)\left(\bar{L}D+\bar{M}\right) \\
&= L\bar{L}D^2 + \left(L\bar{L}' + L\bar{M} + M\bar{L}\right)D + \left(L\bar{M}' + M\bar{M}\right). \qquad (38)
\end{aligned}
$$

Let us consider a peculiar simple situation

$$
L = x(x-1), \qquad \bar{L} = (x-a)(x-b). \qquad (39)
$$

The third relation in (7): $L\bar{M}' + M\bar{M} = Q_2(x)$ obliges M and $\bar{M}$ to be of degree 1. Let $M = xA + B$, $\bar{M} = x\bar{A} + \bar{B}$. The unknown A, B, $\bar{A}$, $\bar{B}$ are given using also the second relation in (7). From $Q_2(x) = \bar{A}x(x-1) + (xA+B)(x\bar{A}+\bar{B})$, we get

$$
\begin{cases}
\bar{A} + A\bar{A} &= \alpha\beta, \\
-\bar{A} + \bar{B}A + B\bar{A} &= \rho_1.
\end{cases} \qquad (40)
$$

As in the Lamé's case, some kind of symmetry appears when permuting L and $\bar{L}$ in the factorization, allowing to reduce by a factor 2 the number of factorizations. Use of the adjoint operator $\mathcal{H}^\star$ of $\mathcal{H}$ is now useful. Indeed, the Lagrange adjoint $\mathcal{H}^\star$ of $\mathcal{H}$ is given by [1,21]

$$
\mathcal{H}^\star = Q^\star_{k+2}D^2 + Q^\star_{k+1}D + Q^\star_k \qquad (41)
$$

with

$$
Q^\star_{k+2} = Q_{k+2}, \quad Q^\star_{k+1} = 2Q'^\star_{k+2} - Q_{k+1}, \quad Q^\star_k = Q''{}_{k+2} - Q'_{k+1} + Q_k \qquad (42)
$$

It is not very easy to write in general $\mathcal{H}^\star$ in the form of the equation (11) with star parameters $\alpha^\star$, $\beta^\star$, $\rho^\star_i$, because the corresponding equations are quadratic. But from the factorized form of $\mathcal{H}$, computation of polynomials $M^\star$ and $\bar{M}^\star$ are trivial from the relations:

$$
\mathcal{H} = (LD+M)(\bar{L}D+\bar{M}), \quad \mathcal{H}^\star = (\bar{L}D+\bar{M})^\star(LD+M)^\star \qquad (43)
$$

and, from equation (42), we get

$$\begin{aligned}\mathcal{H}^\star &= (-\bar{L}D + \bar{L}' - \bar{M})(-LD + L' - M)\\ &= (\bar{L}D + \bar{M}^\star)(LD + M^\star)\end{aligned} \tag{44}$$

with

$$\bar{M}^\star = \bar{M} - \bar{L}',\ M^\star = M - L' \tag{45}$$

generalizing the relation (15).

In tables 5 to 7, we give, in the 14 situations, the values of M, $\bar{M}$, α, β, ρ_1 and ρ_2 as well as those corresponding the other choices of L and $\bar{L}$. Of course, the symmetry in the factorization of $\mathcal{H}_2$ is not so rich as in the case of $\mathcal{L}_2$. Nevertheless, a quick sight in the tables allows to classify families having the same representation for ρ_1 and ρ_2 in terms of $A,\ \bar{A},\ B,\ \bar{B}$. See tables 5 - 7. The symmetry between L and $\bar{L}$ can also be observed in all tables.

6 Concluding remarks

Some important features deserve to be pointed out.

(1) In each case both solutions $y_i(x),\ (i = 1, 2)$ of the factorized equations are easily obtained solving first the equation

$$\bar{L}y_i'(x) + \bar{M}y_i(x) = 0.$$

In all Lamé's cases, these two solutions are given by

$$y_1(x) = \sqrt{\bar{L}(x)},\ y_2(x) = \sqrt{\bar{L}(x)} \int \frac{dx}{\bar{L}(x)\sqrt{L(x)\bar{L}(x)}}. \tag{49}$$

We recover, of course, for $k = 1$, the 8 "pseudo polynomials" of Lamé [22]

$$y(x) = \sqrt{\bar{L}(x)}P_N(x),\ \text{(first type)}, \tag{50}$$

$P_N(x)$ being the corresponding Lamé's polynomials but now with appropriate parameters α, β and ρ_i.

(2) This approach does not give a new help in the search of a pecular solution of a generalized Heun's equation. But it gives a global approach of all solutions for arbitrary parameters $(\epsilon_i,\ \tau_i)$. This factorization defines in some way a parameter space splitted into many domains defined by "factor parameters" given in the several tables.

(3) The factorization becomes more and more difficult to handle with the increase of the number of $k+2$ singularities. Even with $k=3$, there are already 32 different situations to investigate, what is irrelevant to be presented as it implies very long and complicated analytical expressions for the polynomials L, M, $\bar{L}$, $\bar{M}$. A Maple code is now under investigation for symbolic computation for arbitrary number k of singularities.

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References

[1] E. L. Ince, Ordinary differential equation, Dover Publication Inc 1999

[2] T. Kimura, On Fuchsian differential equations reducible to hypergeometric equation by linear transformations, Funkcial. Ekvac. **13** (1970) 213-217.

[3] E. G. C. Poole, Introduction of linear differential equations, Oxford University Press, Oxford Clarenton Press, 1936

[4] F. Bureau, Sur les equations differentielles lineaires du second ordre, Mémoires de la Société Royale des Sciences de Liège, 3^{ie} tome.

[5] G. H. Halphen, Mémoire sur la reduction des equations differentielles lineaires aux formes integrales. Acad. Sc. Paris, 2^{ie} serie t $XXVIII$ 1984.

[6] M. Picard, Traité d'analyse T. III

[7] A. Ronveaux, Editor, Heun's Differential Equations, Oxford University Press, Oxford and New York, 1995.

[8] A. Ronveaux, Factorisation de l'opérateur de Heun, Rev. Questions Sci. **172** (2001) 409-416.

[9] A. Ronveaux, Factorization of the Heun's differential equation, Appl. Math. Comput. **141** (2003) 177-184.

[10] M. N. hounkonnou, A. Ronveaux and K. Sodoga, Factorization of some confluent Heun's differential equations, Appl. Math. Comput. **189** (2007) 816-820.

[11] F. M. Arscott, Heun's Equation, in Heun's Differential Equations (A. Ronveaux, Editor), Oxford University Press, Oxford and New York, 1995, 17-18 and 41-42

[12] R. S. Maier, Private communication (2002)

[13] M. Bronstein, An improved Algorithm for Factoring Linear Ordinary Differential Operators, Proceedings ISSAC'94, Oxford, UK, ACM Press, (1994) 336-340.

[14] F. Schwarz, A factorizzation Algorithm for Linear Ordinary Differential Equations, Proceedings of ISSAC'89, Oxford, ACM Press, (1989) 17-25.

[15] S. P. Tsarev, An algorithm for Complete Enumeration of All Factorization of a Linear Ordinary Differential Operator, Proceedings of ISSAC'89, ACM Press, (1996) 176–189.

[16] M. Van Hoeij, Factorization of Differential Operators with Rational Functions Coeficients, J. Symbolic Computation **24** (1997) 537-561.

[17] A. Pham Ngoc Dinh, Sur les fonctions généralisées de Lamé d'ordre $(n-2)$ associées à une représentation induite de $E(n)$, Bull. Sc. Math. 2^e, série **104** (1980) 113-134.

[18] T. Guttmann, Private communication (1997).

[19] E. S. Cheb-Terrab, New closed form solutions in terms of ${}_pF_q$ for families of the General, Confluent and Bi-confluent Heun differential equations, J. Phys. A: Math.Gen. **37** (2004) 9923

[20] D. Basic and H. Schmid, Heun equation, Teukolsky equation and type-D metrics, J. Math. Phys. **48** (2007) 042502.

[21] G. Birkhoff and G-R. Rota, Ordinary differential equations, Ginn and Company (1962).

[22] G. Petiau, Les classes de solutions de Lamé, International Congress of Mathematics, Helsinki, (August 1978).

Table 2. Factorization of the Lame's operator $\mathcal{L}_2[y]$

$$\mathcal{L}_2[y] = Q_4(x)y'' + Q_3(x)y' + (\alpha\beta x^2 + \rho_1 x + \rho_2)y = 0 = (LD + M)(\bar{L}D + \bar{M})y. \quad M = x + B,\ \bar{M} = -x + \bar{B}.$$

L	$\bar{L}$	B	$\bar{B}$	α	β	ρ_1	ρ_2
$x(x-1)$	$(x-a)(x-b)$	$-\frac{1}{2}$	$\frac{1}{2}(a+b)$	-1	2	$\frac{1}{2}(a+b+3)$	$-\frac{1}{4}(a+b)$
$(x-a)(x-b)$	$x(x-1)$	$-\frac{1}{2}(a+b)$	$\frac{1}{2}$	2	-1	$\frac{1}{2}(3a+3b+1)$	$-\frac{1}{4}(a+b+ab)$
$x(x-a)$	$(x-1)(x-b)$	$-\frac{1}{2}a$	$\frac{1}{2}(b+1)$	-1	2	$\frac{1}{2}(3a+b+1)$	$-\frac{1}{4}(a+ab)$
$(x-1)(x-b)$	$x(x-a)$	$-\frac{1}{2}(b+1)$	$\frac{1}{2}a$	2	-1	$\frac{1}{2}(a+3b+3)$	$-\frac{1}{4}(a+4b+ab)$
$x(x-b)$	$(x-1)(x-a)$	$-\frac{1}{2}b$	$\frac{1}{2}(a+1)$	-1	2	$\frac{1}{2}(a+3b+1)$	$-\frac{1}{4}(b+ab)$
$(x-1)(x-a)$	$x(x-b)$	$-\frac{1}{2}(a+1)$	$\frac{1}{2}b$	2	-1	$\frac{1}{2}(3a+b+3)$	$-\frac{1}{4}(4a+b+ab)$

Table 3. Factorization of the Lame's operator $\mathcal{L}_2[y]$ (continuation).

$$\mathcal{L}_2[y] = Q_4(x)y'' + Q_3(x)y' + (\alpha\beta x^2 + \rho_1 x + \rho_2)y = 0 = (LD + M)(\bar{L}D + \bar{M})y. \quad M = \frac{3}{2}x^2 + Bx + C,\ \bar{M} = -\frac{1}{2},\ \alpha = \frac{3}{2},\ \beta = -\frac{1}{2}.$$

L	$\bar{L}$	B	C	ρ_1	ρ_2
$(x-1)(x-a)(x-b)$	x	$-(a+b+1)$	$\frac{1}{2}(a+b+ab)$	$\frac{1}{2}(a+b+1)$	$-\frac{1}{4}(a+b+ab)$
$x(x-a)(x-b)$	$x-1$	$-(a+b)$	$\frac{1}{2}ab$	$\frac{1}{2}(a+b)$	$-\frac{1}{4}ab$
$x(x-1)(x-b)$	$x-a$	$-(b+1)$	$\frac{1}{2}b$	$\frac{1}{2}(b+1)$	$-\frac{1}{4}b$
$x(x-a)(x-1)$	$x-b$	$-(a+1)$	$\frac{1}{2}a$	$\frac{1}{2}(a+1)$	$-\frac{1}{4}a$

Table 4. Factorization of the Lame's operator $\mathcal{L}_2[y]$ (end).

$$\mathcal{L}_2[y] = Q_4(x)y'' + Q_3(x)y' + (\alpha\beta x^2 + \rho_1 x + \rho_2)y = 0 = (LD + M)(\bar{L}D + \bar{M})y. \quad \bar{M} = -\frac{3}{2}x^2 + \bar{B}x + \bar{C},\ M = \frac{1}{2},\ \alpha = -\frac{3}{2},\ \beta = \frac{5}{2}.$$

L	$\bar{L}$	$\bar{B}$	$\bar{C}$	ρ_1	ρ_2
x	$(x-1)(x-a)(x-b)$	$a+b+1$	$-\frac{1}{2}(a+b+ab)$	$\frac{3}{2}(a+b+1)$	$-\frac{1}{4}(a+b+ab)$
$x-1$	$x(x-a)(x-b)$	$a+b$	$-\frac{1}{2}ab$	$\frac{3}{2}(2+a+b)$	$-\frac{1}{4}(4a+4b+ab)$
$x-a$	$x(x-1)(x-b)$	$b+1$	$-\frac{1}{2}b$	$\frac{3}{2}(2a+b+1)$	$-\frac{1}{4}(4a+b+4ab)$
$x-b$	$x(x-a)(x-1)$	$a+1$	$-\frac{1}{2}a$	$\frac{3}{2}(a+2b+1)$	$-\frac{1}{4}(a+4b+4ab)$

Table 5. Factorization of the Heun's operator $\mathcal{H}_2[y]$

$$\mathcal{H}_2[y] = Q_4(x)y'' + Q_3(x)y' + (\alpha\beta x^2 + \rho_1 x + \rho_2)y = 0 = (LD + M)(\bar{L}D + \bar{M})y. \quad M = Ax + B,\ \bar{M} = \bar{A}x + \bar{B}. \tag{46}$$

L	$\bar{L}$	A	$\overline{A}$	B	$\overline{B}$	α	β
$x(x-1)$	$(x-a)(x-b)$	$\delta+\gamma$	$-2+\epsilon_1+\epsilon_2$	$-\gamma$	$a-\epsilon_2 a+b-\epsilon_1 b$	$-2+\epsilon_1+\epsilon_2$	$\gamma+\delta+1$
$(x-a)(x-b)$	$x(x-1)$	$\epsilon_1+\epsilon_2$	$-2+\gamma+\delta$	$-\epsilon_2 a-\epsilon_1 b$	$1-\gamma$	$\epsilon_1+\epsilon_2+1$	$-2+\gamma+\delta$
$x(x-a)$	$(x-1)(x-b)$	$\epsilon_1+\gamma$	$\delta-2+\epsilon_2$	$-\gamma a$	$1-\epsilon_2-\delta b+b$	$\delta-2+\epsilon_2$	$\gamma+\epsilon_1+1$
$(x-1)(x-b)$	$x(x-a)$	$\delta+\epsilon_2$	$-2+\epsilon_1+\gamma$	$-\delta b-\epsilon_2$	$a-\gamma a$	$1+\delta+\epsilon_2$	$-2+\epsilon_1+\gamma$
$x(x-b)$	$(x-1)(x-a)$	$\gamma+\epsilon_2$	$\delta-2+\epsilon_1$	$-\gamma b$	$1-\epsilon_1-\delta a+a$	$\delta-2+\epsilon_1$	$1+\gamma+\epsilon_2$
$(x-1)(x-a)$	$x(x-b)$	$\epsilon_1+\delta$	$\epsilon_2-2+\gamma$	$-\delta a-\epsilon_1$	$b-\gamma b$	$1+\epsilon_1+\delta$	$\epsilon_2-2+\gamma$

L	$\bar{L}$	ρ_1	ρ_2
$x(x-1)$	$(x-a)(x-b)$	$-\bar{A}+B\bar{A}+A\bar{B}$	$B\bar{B}$
$(x-a)(x-b)$	$x(x-1)$	$-(a+b)\bar{A}+B\bar{A}+A\bar{B}$	$ab\bar{A}+B\bar{B}$
$x(x-a)$	$(x-1)(x-b)$	$-a\bar{A}+B\bar{A}+A\bar{B}$	$B\bar{B}$
$(x-1)(x-b)$	$x(x-a)$	$-(1+b)\bar{A}+B\bar{A}+A\bar{B}$	$b\bar{A}+B\bar{B}$
$x(x-b)$	$(x-1)(x-a)$	$-b\bar{A}+B\bar{A}+A\bar{B}$	$B\bar{B}$
$(x-1)(x-a)$	$x(x-b)$	$-(1+a)\bar{A}+B\bar{A}+A\bar{B}$	$a\bar{A}+B\bar{B}$

Table 6. Factorization of the Heun's operator $\mathcal{H}_2[y]$ (continuation).

$$\mathcal{H}_2[y] = Q_4(x)y'' + Q_3(x)y' + (\alpha\beta x^2 + \rho_1 x + \rho_2)y = 0 = (LD+M)(\bar{L}D+\bar{M})y. \quad M = Ax^2 + Bx + C,\ \bar{M} = \bar{A}. \tag{47}$$

L	$\bar{L}$	A	B	C	$\bar{A}$	α	β	ρ_1	ρ_2
$(x-1)(x-a)(x-b)$	x	$\epsilon_1+\delta+\epsilon_2$	$-\delta a-\delta b-\epsilon_2-\epsilon_2 a-\epsilon_1-\epsilon_1 b$	$\epsilon_2 a+\epsilon_1 b+\delta ab$	$-1+\gamma$	$\epsilon_1+\delta+\epsilon_2$	$-1+\gamma$	$B\bar{A}$	$C\bar{A}$
$x(x-a)(x-b)$	$x-1$	$\epsilon_1+\gamma+\epsilon_2$	$-\epsilon_2 a-\gamma a-\epsilon_1 b-\gamma b$	γab	$-1+\delta$	$\epsilon_1+\gamma+\epsilon_2$	$-1+\delta$	$B\bar{A}$	$C\bar{A}$
$x(x-1)(x-b)$	$x-a$	$\delta+\gamma+\epsilon_2$	$-\epsilon_2-\gamma-\gamma b-\delta b$	γb	$-1+\epsilon_1$	$\delta+\gamma+\epsilon_2$	$-1+\epsilon_1$	$B\bar{A}$	$C\bar{A}$
$x(x-a)(x-1)$	$x-b$	$\epsilon_1+\delta+\gamma$	$-\gamma a-\delta a-\gamma-\epsilon_1$	γa	$-1+\epsilon_2$	$\epsilon_1+\delta+\gamma$	$-1+\epsilon_2$	$B\bar{A}$	$C\bar{A}$

Table 7. Factorization of the Heun's operator $\mathcal{H}_2[y]$ (end).

$$\mathcal{H}_2[y] = Q_4(x)y'' + Q_3(x)y' + (\alpha\beta x^2 + \rho_1 x + \rho_2)y = 0 = (LD + M)(\bar{L}D + \bar{M})y. \quad \bar{M} = \bar{A}x^2 + \bar{B}x + \bar{C},\ M = A. \tag{48}$$

L	$\bar{L}$	A	$\bar{A}$	$\bar{B}$	$\bar{C}$
x	$(x-1)(x-a)(x-b)$	γ	$-3+\epsilon_1+\delta+\epsilon_2$	$2a+2b+2-\delta a-\delta b-\epsilon_2-\epsilon_2 a-\epsilon_1-\epsilon_1 b$	$-ab-a-b+\epsilon_2 a+\epsilon_1 b+\delta ab$
$x-1$	$x(x-a)(x-b)$	δ	$\gamma-3+\epsilon_1+\epsilon_2$	$2b-\epsilon_1 b-\gamma b+2a-\gamma a-\epsilon_2 a$	$-ab+\gamma ab$
$x-a$	$x(x-1)(x-b)$	ϵ_1	$-3+\delta+\gamma+\epsilon_2$	$2b-\delta b-\gamma b+2-\gamma-\epsilon_2$	$-b+\gamma b$
$x-b$	$x(x-a)(x-1)$	ϵ_2	$-3+\epsilon_1+\gamma+\delta$	$2-\epsilon_1-\gamma+2a-\gamma a-\delta a$	$-a+\gamma a$

L	$\bar{L}$	α	β	ρ_1	ρ_2
x	$(x-1)(x-a)(x-b)$	$-3+\epsilon_1+\delta+\epsilon_2$	$2+\gamma$	$\bar{B}+A\bar{B}$	$A\bar{C}$
$x-1$	$x(x-a)(x-b)$	$\gamma-3+\epsilon_1+\epsilon_2$	$2+\delta$	$-2\bar{A}+\bar{B}+A\bar{B}$	$-\bar{B}+A\bar{C}$
$x-a$	$x(x-1)(x-b)$	$-3+\delta+\gamma+\epsilon_2$	$2+\epsilon_1$	$-2a\bar{A}+\bar{B}+A\bar{B}$	$-a\bar{B}+A\bar{C}$
$x-b$	$x(x-a)(x-1)$	$-3+\epsilon_1+\gamma+\delta$	$2+\epsilon_2$	$-2b\bar{A}+\bar{B}+A\bar{B}$	$-b\bar{B}+A\bar{C}$